ON THE FRACTIONAL DIFFERENTIATION OF A FUNCTION OF SEVERAL VARIABLES

BY

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1. In [5] a characterization for fractional differentiation of a function of a real variable is given. Here, the results are extended to the case of a function of several variables.

Before we state these results we must review some definitions. By $x, y, t, \ldots$ we denote points $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n), \ldots$ of n-dimensional Euclidean space. We will consider integrable functions, $f: E_n \to R$, where R is the set of real numbers. We denote the measure of a measurable set E by $|E|$. The symbols $x+y$ and λx, where λ is a scalar, have the usual meaning. By $|x|$ we mean $(\sum_{i=1}^n x_i^2)^{1/2}$, by $\langle x, y\rangle$ we mean $\sum_{i=1}^n x_i y_i$, and if $j=(j_1, j_2, \ldots, j_n)$ where the j_i are integers we use the notation $|j|$ to mean $j_1+j_2+\cdots+j_n$. By x^j we mean $x_1^{j_1}x_2^{j_2}\cdots x_n^{j_n}$. Σ denotes the unit sphere of E_n, σ, μ denote elements of Σ, and $d\sigma$ the usual area element of Σ.

We will use the symbol C, sometimes with subscripts and sometimes without, for an absolute constant or a constant dependent only on the dimension and the parameters of the problem.

By C_0^k we denote the class of functions with compact support and k continuous derivatives, and we write $(\partial/\partial x)^j$ for $(\partial^{j_1}/\partial x_1^{j_1})(\partial^{j_2}/\partial x_2^{j_2})\cdots(\partial^{j_n}/\partial x_n^{j_n})$ and $(\partial/\partial x)^j f = f_j$.

For a function f, we denote its βth integral by f_β and write

$$f_\beta(x) = \int_{E_n} \frac{f(t)\,dt}{|x-t|^{n-\beta}} \qquad (0 < \beta < 1).$$

Whenever the region of integration for an integral is E_n we may simply write $\int$ for $\int_{E_n}$.

A function defined in a neighborhood of a point x_0 is said to have a k derivative if

$$f(x_0+t) = P_{x_0}(t) + R_{x_0}(t) \tag{1}$$

where $P_{x_0}(t)$ is a polynomial in the variable t of degree $\leqq k$ and $R(t) = R_{x_0}(t) = O(|t|^k)$ as $|t| \to 0$. For $1 \leqq p < \infty$, f is said to have a derivative of order k in the L^p sense if $\{\rho^{-n}\int_{|t|\leqq\rho}|R(t)|^p\,dt\}^{1/p} = o(\rho^k)$ as $\rho \to 0$.

For an integer $k \geqq 0$, f is said to have a derivative of order α at x_0, $k<\alpha<k+1$, if f_β has a $k+1$ derivative at x_0 $(\alpha+\beta=k+1)$. f is said to have an α derivative at x_0 in the L^p sense if f_β has a $k+1$ derivative at x_0 in the L^p sense.

Received by the editors March 30, 1967.

Furthermore, with $R(t)$ defined as in (1), f is said to satisfy Λ_α at x_0 if $R(t) = O(|t|^\alpha)$ as $|t| \to 0$. In this case we write $f \in \Lambda_\alpha$. The corresponding L^p definition, Λ_α^p, is satisfied if

$$(2) \qquad \left\{\frac{1}{\rho^n}\int_{|t|<\rho} |R(t)|^p \, dt\right\}^{1/p} = O(\rho^\alpha).$$

Finally f is said to satisfy the condition N_α^p if

$$(3) \qquad \int_{|t|\leqq\rho} \frac{|R(t)|^p}{|t|^{n+p\alpha}} \, dt < \infty \qquad \text{for some } \rho > 0.$$

The main theorems are as follows.

THEOREM 1. *Suppose f satisfies the condition Λ_α at every point of a measurable set E, E has positive measure. Then the necessary and sufficient condition for f to have a derivative of order α almost everywhere in E is that f satisfy the condition N_α^2 almost everywhere in E.*

The condition N_α^2 is the characterizing notion for the fractional differentiability of functions, and if Λ_α is relaxed to Λ_α^2 we have:

THEOREM 2. *f satisfies the condition N_α^2 almost everywhere in the set E if and only if f satisfies the condition Λ_α^2 and f has an α derivative in the L^2 sense almost everywhere in E.*

That N_α^2 is the characterizing notion for differentiability is perhaps underlined by:

THEOREM 2′. *f has an α derivative in the sense of L^p $(2 \leqq p < \infty)$ and satisfies the condition Λ_α^p almost everywhere in a set E if and only if f satisfies both N_α^2 and N_α^p almost everywhere in the set E.*

The proofs of these results rely heavily on some theorems first proved in [4] for the one-dimensional case and then recently extended to the n-dimensional case in [3]. To state these results we make some further remarks.

Suppose that f has a $k-1$ derivative at x_0 in either the ordinary sense or the L^p sense and the polynomial for this case as corresponds to (1) is $P_{x_0}(t)$. Write $\Delta_{x_0}^k(t) = R_{x_0}(t) + (-1)^{k-1}R_{x_0}(-t)$. f is said to satisfy the condition Λ_k^* at x_0 if $\Delta_{x_0}^k(t) = O\{|t|^k\}$ as $|t| \to 0$ and it is said to satisfy the condition N_k^p if there is a $\rho > 0$ such that

$$\int_{|t|\leqq\rho} \frac{|\Delta_{x_0}^p(t)|^p}{|t|^{n+pk}} \, dt < \infty \qquad (1 \leqq p < \infty).$$

The results we speak of are given below.

THEOREM A. *Suppose f satisfies the condition Λ_k^* at every point of a set E. Then f has a k derivative at almost every x in E if and only if f satisfies N_k^2 almost everywhere in E.*

THEOREM B. *The necessary and sufficient condition for f to have a k derivative in the L^p sense* $(2 \leqq p < \infty)$ *almost everywhere in a set E is that f satisfies the conditions* N_k^p, N_k^2 *almost everywhere in E.*

Theorem B is not proved in [3] but this result can be obtained in much the same way as it is done in the one-dimensional case in [4].

The proof of Theorem 2′ does not differ from the proof of Theorem 2 and we will only fully prove Theorem 1 and Theorem 2. The existence of the α derivative at x_0 of f depends on its local properties and hence altering f outside a neighborhood of x_0 does not change the existence of the α derivative.

2. In addition to the above results several lemmas are used in the proofs.

LEMMA 1. (*See* [2, *p.* 148] *for a similar case.*) *Let* $f(x, y)$ *be a measurable function on the product of two measure spaces* $M_1 \times M_2$ *with the measures* μ_1 *and* μ_2 *respectively. Then using the usual notation for* L^p *norms,* $1 \leqq p < \infty$, $\|\int f(\cdot, y)\, d\mu_2\|_p \leqq \int \|f(\cdot, y)\|_p\, d\mu_2$ *where the* L^p *norm is taken in the variable x.*

LEMMA 2. *Suppose that k is an integer,* $0 \leqq k < \alpha < k+1$, *and* $\alpha + \beta = k+1$. *Let g be defined by*

$$g(x) = \int_{E_n} \frac{G_\sigma(x+t)}{|t|^{n+\beta-1}}\, dt$$

where $\sigma = t/|t|$, G_σ *is the direction derivative of G in the direction of* σ, *and* $G \in C_0^{k+1}$. *Then g satisfies the condition* Λ_α *uniformly and g satisfies* N_α^2 *and* N_α^p *for almost all x in* E_n, $2 \leqq p < \infty$.

Proof. Let μ be a unit vector in E_n and by $g_{\mu^k}(x)$ denote the direction derivative of g of order k in the direction of μ. Consider

$$\begin{aligned} g_{\mu^k}(x+\mu\tau) - g_{\mu^k}(x) = \Delta_g^k(x, \mu, \tau) &= \int_{E_n} G_{\sigma,\mu^k}(x+t+\mu\tau) - G_{\sigma,\mu^k}(x+t)\, \frac{dt}{|t|^{n+\beta-1}} \\ &= \int_\Sigma d\sigma \int_0^\infty (G_{\sigma,\mu^k}(x+r\sigma+\tau\mu) - G_{\sigma,\mu^k}(x+r\sigma))\, \frac{dr}{r^\beta} \\ &= \int_\Sigma d\sigma \int_0^\infty (G_{\sigma,\mu^k}(x+r\sigma+\tau\mu) - G_{\sigma,\mu^k}(x+\tau\sigma+r\sigma))\, \frac{dr}{r^\beta} \\ &\quad + \int_\Sigma d\sigma \int_0^\infty (G_{\sigma,\mu^k}(x+\tau\sigma+r\sigma) - G_{\sigma,\mu^k}(x+r\sigma))\, \frac{dr}{r^\beta} \\ &= I_1 + I_2. \end{aligned}$$

$$\begin{aligned} |I_2| &= \left| \int_\Sigma d\sigma \left[\int_0^\tau G_{\sigma,\mu^k}(x+s\sigma)\, \frac{ds}{s^\beta} + \int_\tau^\infty G_{\sigma,\mu^k}(x+s\sigma) \left\{ \frac{1}{|s-\tau|^\beta} - \frac{1}{s^\beta} \right\} ds \right] \right| \\ &\leqq |\Sigma|\, \|G_{\sigma,\mu^k}\|_\infty \left| \int_0^\infty \frac{ds}{|s-\tau|^\beta - s^\beta} \right| \leqq C|\tau|^{1-\beta}; \end{aligned}$$

$$|I_1| = \int_\Sigma d\sigma \left\{ \int_\tau^\infty + \int_0^\tau \Delta^k G_\sigma(x, \mu, \tau, r)\, \frac{dr}{r^\beta} \right\}$$

where $\Delta^k G_\sigma = \Delta^k G_\sigma(x, \mu, \tau, r)$ is the integrand of I_1. The first of these integrals can be estimated by first integrating by parts and noting that G_{μ^k} is Lipschitz. The second integral can be estimated by replacing $\Delta^k G_\sigma$ by its maximum and integrating. Hence $|I_2| \leqq C|\tau|^{1-\beta}$. It only remains to apply Taylor's theorem to g to see that g is in Λ_α, with the polynomial in (1) for g being its Taylor's development of order k.

Next, to show g satisfies the condition N_α^2, define the function $\omega_{x,\mu}(t)$ by

$$\omega_{x,\mu}(t)|t|^{1-\beta} = |g_{\mu^k}(x+t) - g_{\mu^k}(x)|$$

with $\mu = t/|t|$. We first show

$$\int_{E_n} \frac{|\omega_{x,\mu}(t)|^2}{|t|^n} dt < \infty. \tag{4}$$

To do this integrate (4) with respect to x. After an application of Parseval's equality we have

$$\frac{1}{(2\pi)^n} \int_{E_n} \left\{ \int_{E_n} \frac{|\omega_{x,\mu}(t)|^2}{|t|^n} dt \right\} dx = \int_{E_n} |t|^{2(\beta-1)-n} dt \int_{E_n} 2\sin^2 \frac{\langle y, t\rangle}{2} |\widehat{g_{\mu^k}}(y)|^2 dy \tag{5}$$

where $\widehat{g_{\mu^k}}(y) = \int_{E_n} g_{\mu^k}(x) \cdot e^{-i\langle x,y\rangle} dx$. The integral

$$\int_{E_n} |t|^{2(\beta-1)-n} |y|^{2(1-\beta)} 2\sin^2(\langle y, t\rangle/2)\, dt \leqq C$$

where C is a constant depending on only β and n. The remaining integral is equal to the square of

$$\left\{ \int_{E_n} \left[\int_\Sigma d\sigma \int_0^\infty e^{-i\langle y, r\sigma\rangle} \widehat{G_{\sigma,\mu^k}}(y) \cdot |y|^{\beta-1} \frac{dr}{r^\beta} \right]^2 dy \right\}^{1/2}$$

$$= \left\{ \int_{E_n} \left[\int_\Sigma G_{\sigma,\mu^k}(y) \cdot |y|^{\beta-1} d\sigma \int_0^\infty e^{-i\langle y, r\sigma\rangle} \frac{dr}{r^\beta} \right]^2 dy \right\}^{1/2}.$$

But $\left|\int_0^\infty e^{-i\langle y, r\sigma\rangle} r^{-\beta} dr\right| = \text{constant} \cdot |\langle y, \sigma\rangle|^{1-\beta} < C|y|^{1-\beta}$. Using this and applying Lemma 1 we have (5) is $\leqq C \int_\Sigma \{\int_{E_n} |\widehat{G_{\sigma,\mu^k}}(y)|^2 dy\}^{1/2} d\sigma \leqq C|\Sigma|\, \|G_{\sigma,\mu^k}\|_2 < \infty$.

Now let x_0 be a point where (4) is finite. Writing $g(x_0+t) = P_{x_0}(t) + R_{x_0}(t)$ where $P_{x_0}(t)$ is the usual polynomial in the development of g we see that

$$|R_{x_0}(t)| = \frac{1}{(k-1)!} \left| \int_0^{|t|} (|t|-u)^{k-1} \omega_{x_0,\sigma}(u\sigma) |u|^{1-\beta} du \right|$$

where $\sigma = t/|t|$. Hence

$$\int_{E_n} \frac{|R_{x_0}(t)|^2}{|t|^{n+2\alpha}} dt = \int_\Sigma d\sigma \int_0^\infty \frac{|R_{x_0}(r\sigma)|^2}{r^{1+2\alpha}} dr$$

$$\leqq C \int_\Sigma d\sigma \int_0^\infty \left\{ \int_0^r r^{k-\beta} |\omega_{x_0,\sigma}(u\sigma)|\, du \right\}^2 \frac{dr}{r^{1+2\alpha}}$$

$$= C \int_\Sigma d\sigma \int_0^\infty \left\{ \frac{1}{r} \int_0^r |\omega_{x_0,\sigma}(u\sigma)|\, du \right\}^2 \frac{dr}{r}.$$

After an application of Hölder's inequality and a change of order of integration of the inner two integrals this is less than or equal to a constant times

$$\int_{\Sigma} d\sigma \int_0^{\infty} |\omega_{x_0,\sigma}(u\sigma)|^2 \left\{\int_u^{\infty} \frac{dr}{r^2}\right\} du = \int_{E_n} \frac{|\omega_{x_0,\sigma}(t)|^2}{t^n} dt < \infty.$$

For the case $2 \leqq p < \infty$ let A be an upper bound for $\omega_{x_0}(t)$. Then one has

$$\int_{E_n} \frac{|\omega_{x_0}(t)|^p}{|t|^n} dt \leqq \frac{A^p}{A^2} \int_{E_n} \frac{|\omega_{x_0}(t)|^2}{|t|^n} dt < \infty.$$

LEMMA 3. *Suppose that $f \in L^2(E_n)$ and has finite support, and that $F(x) = f_\beta(x)$, $0 < \beta < 1$. Then*

$$\frac{1}{A_\beta} \int_{|t| \geqq \varepsilon} \frac{F(x+t) - F(x)}{|t|^{n+\beta}} dt$$

converges to $f(x)$ in the L^2 norm as $\varepsilon \to 0$ for a suitable choice of A_β. A_β is a nonzero constant depending only on β and n.

Proof. The Fourier transform of $F(x+t) - F(x)$ is $B_\beta f^\wedge(x)\{e^{i\langle x,t\rangle} - 1\}|x|^{-\beta}$ where B_β is the constant such that $\{1/|x|^{n-\beta}\}^\wedge = B_\beta |x|^{-\beta}$. Therefore the Fourier transform of $\int_{|t| \geqq \varepsilon} ((F(x+t) - F(x))/|t|^{n+\beta})\, dt = f^\wedge(x) M_\varepsilon(x)$ where

$$M_\varepsilon(x) = B_\beta |x|^{-\beta} \int_{|t| \geqq \varepsilon} \frac{e^{i\langle x,t\rangle} - 1}{|t|^{n+\beta}} dt.$$

It is easy to see that $M_\varepsilon(x)$ is bounded in x and ε uniformly for $x \neq 0$ and that the limit of $M_\varepsilon(x)$ as $\varepsilon \to 0$ is nonzero. An application of Plancherel's theorem gives the required result.

LEMMA 4. (*See* [1, *p.* 184].) *Let P be a closed set and $U = \{x : d(x, P) < 1\}$. There exists a covering of $U - P$ by nonoverlapping cubes K with the property that* diam $K \leqq d(P, K) \leqq 3$ diam K. *$d(x, P)$ and $d(P, K)$ represent the distance from x to P and from P to K respectively and* diam *K represents the diameter of the cube K.*

Here the conclusion has been slightly altered from that of [1] but it is not essentially different in the proof.

LEMMA 5. (*See* [6, *p.* 130].) *Let P be a closed set and U be as above. Set $\Delta(x) = d(x, P)$ for x in U and zero otherwise. Then for $\lambda > 0$ we have*

$$\int \frac{\Delta^\lambda(x_0 + t)}{|t|^{n+\lambda}} dt < \infty$$

for almost all $x_0 \in P$.

LEMMA 5′. (*See* [1, *p.* 189].) *Suppose that $\lambda > 0$ and*

$$\frac{1}{h^n} \int_{|t| < h} H(x_0 + t)\, dt \leqq Ah^\lambda, \qquad 0 < h < \infty,$$

for every x_0 in P. Then $\int (H(x_0 + t)/|t|^{n+\lambda})\, dt < \infty$ for almost all x_0 in P.

LEMMA 5″. (*See* [6, *p.* 131].) *Let* K_μ *be the sets of the cover in Lemma* 4. *Then*

$$\sum_\mu \frac{(\operatorname{diam} K_\mu)^{n+\lambda}}{|x_0-x_\mu|^{n+\lambda}} < \infty \qquad \text{for almost all } x_0 \text{ in } P.$$

Here x_μ *is a point of* P *such that* $d(x_\mu, K)=d(P, K)$ *and* $\lambda>0$.

LEMMA 6. (*See* [1, *p.* 183].) *Suppose* $\alpha>0$, *with* $k<\alpha<k+1$, *and that* h *has the development* $h(x_0+t)=\sum_{|j|=0}^{k} h_j(x_0)t^j+R_{x_0}(t)$ *for each* x_0 *in* P, *with* $(1/\rho^n)\int_{|t|\leqq\rho}|R_{x_0}(t)|^p\,dt\leqq A\rho^\alpha$ *for* $0<\rho\leqq\delta$. *Then with* x_0 *in* P *and* x_0+t *in* P,

$$h_e(x_0+t) = \sum_{|j|=0}^{k-|e|}\frac{t^j}{j!}h_{j+e}(x_0)+O(|t|^{\alpha+|e|})$$

for $|e|=0, 1,\dots,k$; O *is uniform for* x_0, x_0+t *in* P, *and* $|t|\leqq\delta$.

LEMMA 7. (*See* [1, *p.* 189].) *Suppose that* F *has a* $k+1$ *derivative in the* L^2 *sense on a set* E, $|E|>0$, *uniformly. Also assume* F *has finite support. Then there exists a function* $G\in C_0^{k+1}$ *such that* $F=G+H$ *where* $H(x_0)=0$ *for* x_0 *in* E *and*

$$\frac{1}{\rho^n}\int_{|t|<\rho}|H(x_0+t)|^2\,dt = o(|t|^{2k+2}) \qquad \text{for } x_0 \text{ in } E \text{ uniformly.}$$

3. In this section we show the conditions of Theorem 1 are sufficient for f to have an α derivative almost everywhere in E. As mentioned earlier we will not do the L^p case since it differs from the L^2 case in an unessential way. In view of Theorem A it is enough to show that $F=f_\beta$ satisfies condition Λ^*_{k+1} and N^2_{k+1} almost everywhere in E.

Suppose that x_0 in E is the origin and that f has support contained in the sphere $S_a=\{x : |x|\leqq a\}$. Let $\lambda(t)$ be a function which is infinitely differentiable and has support in the sphere $S_b=\{x : |x|\leqq b\}$ with $b>a$ and $\lambda(t)=1$ for $|t|\leqq a$. Hence $f(t)=\lambda(t)f(t)$. Let $P(t)$ be the polynomial in (1) and $R(t)$ be the remainder. Then $f(t)=\lambda(t)P(t)+\lambda(t)R(t)$. The integral $\int_{E_n}\lambda(t)P(t)|x-t|^{\beta-n}\,dt$ represents an infinitely differentiable function. Hence we can make the simplifying assumptions that x_0 is the origin and that $f(t)=R(t)$ satisfies the condition Λ^2_α at x_0 and has support in S_b. We also assume k even since the case k odd is similar. Assume $0<|h|<b/2$. Then

$$\begin{aligned}\tfrac12\{F(h)+F(-h)\} &= \int_{S_b} R(t)\tfrac12\{|h-t|^{\beta-n}+|h+t|^{\beta-n}\}\,dt\\ &= \int_{S_b}\frac{R(t)+R(-t)}{2}\cdot|h-t|^{\beta-n}\,dt\\ &= \int_{S_b}\omega(t)|t|^\alpha\cdot|h-t|^{\beta-n}\,dt.\end{aligned}$$

We first show the last integral is the sum of a polynomial in h of degree $\leqq k+1$ and a remainder which is $O(|h|^{k+1})$ as $|h|\to 0$.

Split this integral into the two integrals $\int_{|t|\leqq 2|h|}+\int_{2|h|\leqq|t|\leqq b}$. The first is $\leqq O(|h|^{\alpha})\int_{|t|<2|h|}|h-t|^{\beta-n}\,dt\leqq O(|h|^{k+1})$. For the second, expand $|h-t|^{\beta-n}$ and $|h+t|^{\beta-n}$ in their respective Taylor's developments to obtain

$$\int_{2|h|\leqq|t|\leqq b}\frac{R(t)}{2}|t|^{\beta-n}\left\{P\left(\frac{h}{|t|}\right)+O\left(\left|\frac{h}{t}\right|^{k+2}\right)\right\}dt$$

where P is a polynomial of degree $\leqq k+1$ containing even terms only, i.e., a term of $P(x)$ is $a_j x^j$ where $|j|$ is even. Since $R(t)=O(|t|^{\alpha})$

$$\int_{2|h|\leqq|t|\leqq b}O\left(\left|\frac{h}{t}\right|^{k+2}\right)\frac{R(t)}{2}\,dt = O(|h|^{k+1}).$$

For $|j|=0, 2, 4, \ldots, \frac{1}{2}k$ we have

$$\int_{2|h|\leqq|t|\leqq b}\frac{R(t)}{2}|t|^{\beta-n}a_j\frac{h^j}{|t|^{|j|}}\,dt$$

$$= a_j h^j\int_{|t|\leqq b}\frac{R(t)}{2}|t|^{\beta-n-|j|}\,dt+a_j h^j\int_{|t|<2|h|}O(|t|^{\beta-n+-|j|})\,dt$$

$$= a_j h^j\int_{|t|\leqq b}\frac{R(t)}{2}|t|^{\beta-n-|j|}\,dt+O(|h|^{k+1}).$$

Collecting these results we see we have F equal to a polynomial of degree $\leqq k$ plus a term which is $O(|h|^{k+1})$.

Now it remains to show that if $\eta(t)|t|^{k+1}=F(t)-P(t)$, where $P(t)$ is the polynomial obtained in the above argument, then

$$\int_{|t|\leqq\rho}\frac{\eta^2(t)}{|t|^n}\,dt < \infty$$

for some $\rho>0$. This will be accomplished if we show $\int_{|t|\leqq\rho}(\eta_i^2(t)/|t|^n)\,dt<\infty$, $i=1, 2, 3$, where

$$\eta_1(t) = |t|^{-(k+1)}\int_{|h|\leqq 2|t|}R(h)|h-t|^{\beta-n}\,dh$$

$$\eta_2(t) = |t|\int_{2|t|\leqq|h|\leqq b}R(h)|h|^{-n-\alpha-1}\,dh$$

$$\eta_3(t) = |t|^{-(k+1)+|j|}\int_{|h|\leqq 2|t|}R(h)|h|^{\beta-n-|j|}\,dh$$

where $|j|=0, 2, 4, \ldots, k$.

In each of these cases a similar argument is used. We will do the argument in full for the case of η_1. Let $g(t)$ be a function such that

$$\int_{|t|\leqq\rho}g^2(t)\frac{dt}{|t|^{2(\alpha+\beta)+n}} = 1$$

and assume that $\int_{|h|\leqq 2\rho}(R^2(h)/|h|^{2\alpha+n})\,dh<\infty$. If we can show

$$\int_{|t|\leqq\rho} g(t)\int_{|h|\leqq 2|t|} R(h)|h-t|^{\beta-n}\,dh\,\frac{dt}{|t|^{2(\alpha+\beta)+n}}$$

is finite, this will prove

$$\int_{|t|\leqq\rho}\left\{\int_{|h|\leqq 2|t|} R(h)|h-t|^{\beta-n}\,dh\right\}^2\frac{dt}{|t|^{2(\alpha+\beta)+n}}$$

is finite as required. Rewriting the above using the notation that $t=|t|\sigma$ where σ is the unit vector $t|t|^{-1}$ and $\tau=h|t|^{-1}$ we obtain

$$\int_{0\leqq|t|\leqq\rho} g(t)\frac{dt}{|t|^{2(\alpha+\beta)+n}}\int_{|\tau|\leqq 2}\frac{|t|^{\beta}R(|t|\tau)}{|\tau-\sigma|^{n-\beta}}\,d\tau$$

$$=\int_{0\leqq|t|\leqq\rho;\,|\tau|\leqq 2}\frac{g(t)}{|t|^{\alpha+\beta}}\cdot\frac{R(|t|\tau)}{|t|^{\alpha}}\frac{dt}{|t|^{n}}\cdot\frac{d\tau}{|\tau-\sigma|^{n-\beta}}.$$

Applying Hölder's inequality with the measure $(dt/|t|^n)\cdot(d\tau/|\tau-\sigma|^{n-\beta})$ this is

$$\leqq\left\{\int\!\!\int_{0\leqq|t|\leqq\rho;\,|\tau|\leqq 2}\frac{g^2(t)}{|t|^{2(\alpha+\beta)}}\frac{dt}{|t|^{n}}\frac{d\tau}{|\tau-\sigma|^{n-\beta}}\right\}^{1/2}$$

$$\times\left\{\int\!\!\int_{0\leqq|t|\leqq\rho;\,|\tau|\leqq 2}\frac{R^2(|t|\tau)}{|t|^{2\alpha}}\frac{dt}{|t|^{n}}\frac{d\tau}{|\tau-\sigma|^{n-\beta}}\right\}^{1/2}.$$

The square of the first integral in this product is

$$\leqq\int_{0\leqq|t|\leqq\rho} g^2(t)\frac{dt}{|t|^{2(\alpha+\beta)+n+\beta}}\int_{|h|\leqq 2|t|}\frac{dh}{|h-t|^{n-\beta}}.$$

The inner integral is $\leqq C|t|^{\beta}$ and hence the first term in the product is finite. The square of the second term is equal to

$$\int_{|t|\leqq\rho}\int_{|h|\leqq 2|t|}\frac{R^2(h)}{|h|^{2\alpha+n}}\frac{|\tau|^{2\alpha+n-\beta}|h|^{\beta}}{|h-t|^{n-\beta}}\,dh\,dt$$

$$\leqq(2\rho)^{\beta}(2)^{2\alpha+n-\beta}\int_{|t|\leqq\rho}\int_{|h|\leqq 2\rho}\frac{R^2(h)}{|h|^{2\alpha+n}}\,dh\,\frac{dt}{|h-t|^{n-\beta}}$$

$$\leqq(2\rho)^{\beta}(2)^{2\alpha+n-\beta}\int_{|h|\leqq 2\rho}\int_{|t|\leqq 3\rho}\frac{R^2(h)}{|h|^{2\alpha+n}}\,dh\,\frac{dt}{|t|^{n-\beta}}<\infty.$$

4. We consider the necessity of the conditions of Theorem 1 and Theorem 2 from this point on.

The assumption $(1/\rho^n)\int_{|t|\leqq\rho}|f(x_0+t)-P_{x_0}(t)|^p\,dt=O(\rho^{p\alpha})$, $2\leqq p<\infty$, clearly implies f is locally integrable to the pth power. Thus we may modify f to have finite support and to be in L^p. In addition, we may limit our consideration to a closed set $P\subset E$, where $|E-P|<\varepsilon$ and $\varepsilon>0$ is arbitrary, on which $f_\beta=F$ has a $k+1$ derivative in the L^p sense uniformly and satisfies the condition Λ_α^p uniformly there.

By Lemma 7 we may write $F(x)=G(x)+H(x)$ where $G(x) \in C_0^{k+1}$ and $G_j(x_0) = F_j(x_0)$ represents the jth coefficient of the polynomial in (1). Also we know that $H=F-G$ is zero on P and because of the uniform differentiability of F and the fact that H has compact support we have

$$\int_{|t| \leqq \rho} |H(x_0+t)|^p \, dt \leqq \text{constant } \rho^{p(k+1)+n}, \qquad 0 < \rho < \infty, \tag{6}$$

for x_0 in P.

Apply the inversion formula of Lemma 3 to F to obtain

$$f(x) = \lim_{\varepsilon \to 0} \frac{1}{A_\beta} \int_{|t| \geqq \varepsilon} \frac{F(x+t)-F(x)}{|t|^{n+\beta}} \, dt$$

where the limit is taken in the L^2 norm. Also consider the function $g(x)= \lim_{\varepsilon \to 0} (1/A_\beta) \int_{|t| \geqq \varepsilon} (G(x+t)-G(x)/|t|^{n+\beta}) \, dt$. Since $G \in C_0^{k+1}$ this limit exists in the L^2 norm uniformly in x. Calculating $g(x)$ we find

$$\begin{aligned} g(x) &= \lim_{\varepsilon \to 0} \frac{1}{A_\beta} \int_\Sigma d\sigma \int_\varepsilon^\infty \frac{G(x+r\sigma)-G(x)}{r^{1+\beta}} \, dr \\ &= \lim_{\varepsilon \to 0} \frac{1}{A_\beta} \int_\Sigma d\sigma \int_\varepsilon^\infty \left\{ \int_0^r G_\sigma(x+s\sigma) \, ds \right\} \frac{dr}{r^{1+\beta}}. \end{aligned}$$

After changing the order of integration of the inner integrals and letting $\varepsilon \to 0$ this becomes

$$\frac{1}{\beta A_\beta} \int_\Sigma d\sigma \int_0^\infty G_\sigma(x+s\sigma) \frac{ds}{s^\beta}.$$

Note that $g(x)$ is a function which satisfies the hypothesis of Lemma 2.

Set $h(x)=f(x)-g(x)$. Then we have

$$h(x) = \lim_{\varepsilon \to 0} \frac{1}{A_\beta} \int_{|t| \geqq \varepsilon} \frac{H(x+t)-H(x)}{|t|^{n+\beta}} \, dt$$

in the L^2 norm. However $H(x)=0$ if x in P and the integral $\int_{E_n} (H(x+t)/|t|^{n+\beta}) \, dt$ converges for almost every x in P by Lemma 4′ and (6). By shrinking P further we may assume that $h(x_0)=(1/A_\beta) \int_{E_n} (H(x_0+t)/|t|^{n+\beta}) \, dt$ for all x_0 in P. Since g satisfies the hypothesis of Lemma 2, it satisfies the condition Λ_α uniformly and it satisfies the condition N_α^p, $2 \leqq p < \infty$, almost everywhere. Also f satisfies the condition Λ_α uniformly in P; hence the problem of showing that f satisfies the condition N_α^p almost everywhere in P reduces to showing that h satisfies the condition N_α^p almost everywhere in P. To do this we assume that x_0 is a point of density of P at which the Lemmas 5, 5′, 5″ and 6 hold. We may assume that x_0 is the origin. Since the constant A_β plays no essential role from this point on, we drop it.

Hence we need to show we can write

$$h(x) = P(t)+\xi(t)|t|^\alpha \tag{7}$$

where $P(t)=\sum_{|j|=0}^{k}(a_j t^j/j!)$ and $\int_{|t|\leqq 0}(|\xi(t)|^p/|t|^n)\,dt<\infty$. As before we will do the problem for the case $p=2$ since the case $2\leqq p<\infty$ is similar. If x is in P we can write

$$h(x) = \int_{E_n}\frac{H(t)}{|x-t|^{n+\beta}}\,dt = \int_{|t|\geqq 2|x|}+\int_{|t|\leqq 2x} = S+T.$$

If $|t|\geqq 2|x|$ then $|x-t|^{-(n+\beta)}=|t|^{-(n+\beta)}\{P_t(x)+R\}$ where $P_t(x)$ is the Taylor's development of $|x/|t|-t/|t||^{-(n-\beta)}$ up to and including the kth terms and R is the remainder which is $O(|x/t|^{k+1})$. Thus

$$S = \sum_{|j|=0}^{k}A_j x^j\int_{E_n}\frac{H(t)}{|t|^{|j|}\,|t|^{n+\beta}}\,dt+R_1$$

where

$$R_1 = \sum_{|j|=0}^{k}A_j x^j\int_{|t|\leqq 2|x|}\frac{H(t)}{|t|^{n+\beta+|j|}}\,dt+O\left(|x|^{k+1}\int_{|t|\geqq 2|x|}\frac{|H(t)|}{|t|^{k+1+\beta+n}}\,dt\right).$$

One can easily see that

$$T = \int_{|t|\leqq 2|x|}\frac{|H(t)|}{|t-x|^{n+\beta}}\,dt \leqq \int_{|x/2|\leqq|t|\leqq 2|x|}\frac{|H(t)|}{|t-x|^{n+\beta}}\,dt+\text{constant}\int_{|t|\leqq 2|x|}\frac{|H(t)|}{|x|^{n+\beta}}\,dt.$$

The first integral of S is a polynomial of degree less than or equal to k. To show that h satisfies the condition N_α^2 it is enough to show that

$$\int_{|x|\leqq\delta}\frac{I_i^2(x)\,dx}{|x|^n} < \infty,$$

$i=1, 2, 3, 4$, for some $\delta>0$, where

$$I_1(x) = |x|^\beta\int_{|t|\geqq 2|x|}\frac{|H(t)|}{|t|^{k+1+\beta+n}}\,dt,$$

$$I_2(x) = |x|^{|j|-\alpha}\int_{|t|\leqq 2|x|}\frac{|H(t)|}{|t|^{n+\beta+|j|}}\,dt, \qquad |j| = 0, 1,\dots, k,$$

$$I_3(x) = |x|^{-\alpha}\int_{|x|/2\leqq|t|\leqq 2|x|}\frac{|H(t)|}{|t-x|^{n+\beta}}\,dt,$$

$$I_4(x) = |x|^{-(\alpha+\beta+n)}\int_{|t|\leqq 2|x|}|H(t)|\,dt.$$

To do this we do the integration over the set of points which are in the set P and later we consider the integration over the complement of P.

First consider the above integral with $i=1$. As in an earlier argument suppose that $g(x)$ is such that $\int_{|x|\leqq\delta}(g^2(x)/|x|^n)\,dx=1$. Then we must show $\int_{|x|\leqq\delta}(g(x)I_1(x)/|x|^n)\,dx<\infty$. This integral is equal to

$$\int_{|x|\leqq\delta}\int_{|t|\geqq 2|x|}g(x)|x|^{\beta/2}\frac{|x|^{\beta/2}|H(t)|}{|t|^{k+1}}\frac{dt}{|t|^{n+\beta}}\frac{dx}{|x|^n}$$

$$\leqq\left\{\int_{|x|\leqq\delta}\int_{|t|\geqq 2|x|}g^2(x)|x|^\beta\frac{dt}{|t|^{n+\beta}}\frac{dx}{|x|^n}\right\}^{1/2}\left\{\int_{|x|\leqq\delta}\int_{|t|\geqq 2|x|}\frac{|x|^\beta H^2(t)}{|t|^{2(k+1)}}\frac{dt}{|t|^{n+\beta}}\frac{dx}{|x|^n}\right\}^{1/2}.$$

Changing the order of integration in the second integral of this product and integrating over all x gives

$$\int_{|t|\geqq 0}\int_{|x|\leqq t/2}|x|^{\beta}\frac{H^2(t)}{|t|^{2(k+1)}}\frac{dx}{|x|^n}\frac{dt}{|t|^{n+\beta}}\leqq \text{constant}\int_{|t|\geqq 0}\frac{H^2(t)}{|t|^{2(k+1)+n}}\,dt<\infty.$$

On the other hand, one can easily see the first term in the product is finite.

In a similar way one can obtain the required results in the cases for the integrals I_2 and I_4. For the remainder of this section we concentrate on I_3.

Let $U=\{x : d(x,P)<1\}$. Let $Q=U-P$ and let K_μ be the elements of the cover of Q as given in Lemma 4. Let x be in P and consider the integral $\int_{t\in K_\mu}(|H(t)|/|x-t|^{n+\beta})\,dt$. We have that, the diameter of $K=\operatorname{diam}(K_\mu)$, $\operatorname{diam}(K_\mu)\leqq d(P,K_\mu)\leqq 3\operatorname{diam}(K_\mu)$. For each K_μ we let a_μ be the center of K_μ, and x_μ be a point of P such that $d(x_\mu,K_\mu)=d(P,K_\mu)$. Let S_μ be the smallest sphere containing K_μ with center at x_μ.

Suppose that $d(x,x_\mu)\geqq 4\operatorname{diam}(K_\mu)$. Then

$$\int_{K_\mu}\frac{|H(t)|}{|x-t|^{n+\beta}}\,dt\leqq C|x-x_\mu|^{-(n+\beta)}\left\{\int_{S_\mu}|H(t)|\,dt\right\},$$

since $|x-t|\geqq\frac{1}{4}|x-x_\mu|$. By (6) this is $\leqq C|x-x_\mu|^{-(n+\beta)}\operatorname{diam}(K_\mu)^{k+1+n}$ $=C|x-x_\mu|^{-(n+\beta)}\int_{K_\mu}\Delta^{k+1}(t)\,dt$ where $\Delta(t)$ is the distance of t from P. Since $|x-t|\leqq 3|x-x_\mu|$ the last line is a constant times $\int_{K_\mu}(\Delta^{k+1}(t)/|x-t|^{n+\beta})\,dt$. Suppose that $|x-x_\mu|<4\operatorname{diam}(K_\mu)$. Then

$$\int_{K_\mu}\frac{|H(t)|}{|x-t|^{n+\beta}}\,dt\leqq C\int_{K_\mu}\frac{|H(t)|}{|x_\mu-t|^{n+\beta}}\,dt$$

since $|x-t|\geqq\operatorname{diam}(K_\mu)$ and $|x_\mu-t|\leqq 4\operatorname{diam}(K_\mu)$. We have $|x_\mu-a_\mu|\leqq 4\operatorname{diam}(K_\mu)$ $\leqq 4|x_\mu-t|$ and the above integral is

$$\leqq C\int_{S_\mu}\frac{|H(t)|}{|x_\mu-a_\mu|^{n+\beta}}\,dt\leqq C|x_\mu-a_\mu|^{\alpha}$$

$$\leqq C|x_\mu-a_\mu|^{-(n+\beta)}\int_{K_\mu}\Delta^{k+1}(t)\,dt.$$

One can show $|x_\mu-a_\mu|\geqq|x-t|/8$ for t in K_μ so we find that

$$\int_{K_\mu}\frac{|H(t)|}{|x-t|^{n+\beta}}\,dt\leqq C\int_{K_\mu}\frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}}\,dt\qquad\text{for } x \text{ in } P. \tag{8}$$

Since the origin is a point of density of P and $K_\mu\subset Q$, $\operatorname{diam}(K_\mu)=o(\operatorname{diam}S_\mu(o))$ as $\operatorname{diam}(S_\mu(o))\to 0$ where $S_\mu(o)$ is the smallest sphere containing K_μ with center at the origin. Hence there is a $\delta>0$ such that for $\operatorname{diam}(S_\mu(o))<2\delta$, $\operatorname{diam}(K_\mu)$

$<\operatorname{diam}(S_\mu(o))/6$. If we assume that $|x|<\delta$ then it is easy to see that if $E_x=\{t : |x|/2 \leqq |t| \leqq 2|x|\}$

$$\int_{|x/2|\leqq|t|\leqq 2|x|} \frac{|H(t)|}{|x-t|^{n+\beta}}\,dt \leqq \sum_{K_\mu\cap E_x\neq\varnothing} \int_{K_\mu} \frac{H(t)}{|x-t|^{n+\beta}}\,dt \leqq C \sum_{K_\mu\cap E_x\neq\varnothing} \int_{K_\mu} \frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}}\,dt$$

$$\leqq C\int_{|x/6|\leqq|t|\leqq 4|x|} \frac{\Delta^{k+1}(t)}{|x-t|^{n+\beta}}\,dt.$$

If we choose $\delta<1/4$ the integration is over a region contained in U. Set $\Delta(t)=0$ if $t\notin U$. By applying the estimation technique used on the other integrals one can now show that $\int_{|x|\leqq\delta}(I_3^2(x)/|x|^n)\,dx<\infty$ after noting that x in P and, $\int(\Delta^\lambda(x)/|x|^{\lambda+n})\,dx<\infty$ for $\lambda>0$ by Lemma 5, where one will need to use $\lambda=2(k+1)$.

5. To complete the demonstration that h satisfies the condition N_α^2 one must show that $\int_{Q;\,|x|\leqq\delta}(\xi^2(x)/|x|^n)\,dx<\infty$. Since $\{K_\mu\}$ is a cover for Q the finiteness of this will be proved if we show the two sums

$$\sum_\mu \int_{K_\mu} [\rho(x)-\rho(x_\mu)]^2 \frac{dx}{|x|^{2\alpha+n}} \tag{9}$$

and

$$\sum_\mu \int_{K_\mu} [\rho(x_\mu)]^2 \frac{dx}{|x|^{2\alpha+n}} \tag{10}$$

are finite, where $\xi(x)|x|^\alpha=\rho(x)$.

We consider (10) first. Since x_μ is a point of P, $\xi(x_\mu)$ is majorized by the sum of the integrals I_1, I_2, I_3 and I_4 with $x=x_\mu$. Since the origin is a point of density there is a $\delta>0$ such that for $K_\mu\subset\{x : |x|<\delta\}$ and x in K_μ, $(|x_\mu|/2)\leqq|x|\leqq 2|x_\mu|$ and hence in this case (10) is majorized by a constant times

$$\sum_\mu \int_{K_\mu} \frac{\xi^2(x_\mu)}{|x|^n}\,dx. \tag{11}$$

One sees that, for x in K_μ, $I_1(x_\mu)$ is increased by a constant times

$$|x|^\beta \int_{|t|\geqq|x|} \frac{|H(t)|}{|t|^{k+1+\beta+n}}\,dt.$$

Likewise $I_2(x_\mu)$ and $I_4(x_\mu)$ majorized by constant multiples of

$$|x|^{|j|-\alpha} \int_{|t|\leqq 4|x|} \frac{|H(t)|}{|t|^{n+\beta+|j|}}\,dt, \qquad |j| = 0, 1, \ldots, k,$$

and $|x|^{-(\alpha+\beta+n)}\int_{|t|\leqq 4|x|}|H(t)|\,dt$ respectively. The contribution of these integrals to (11) can be estimated in the same way as in the previous case.

It remains to estimate the contribution of $I_3(x_\mu)$ to the convergence of (11). In view of (8) and the fact that x_μ is in P we have (11) is finite if

$$\sum_\mu \int_{K_\mu} \frac{dx}{|x|^n}\left\{|x_\mu|^{-\alpha}\int_{|x_\mu|/4\leqq|t|\leqq 4|x_\mu|}\frac{|H(t)|}{|t-x_\mu|^{n+\beta}}\,dt\right\}^2$$
$$\leqq C\sum_\mu \int_{K_\mu}\frac{dx}{|x|^{n+\alpha}}\int_{|x|/16\leqq|t|\leqq 16|x|}\frac{\Delta^{k+1}(t)}{|t-x_\mu|^{n+\beta}}\,dt$$

is finite. The last inequality follows by a use of the inequalities $\Delta(t)\leqq|t-x_\mu|$ and $|x_\mu|/2\leqq|x|\leqq 2|x_\mu|$. The limits of integration can be refined.

Let S'_μ be the sphere with center at x_μ and radius twice that of S_μ. It is easy to see that $\int_{S'_\mu}(\Delta^{k+1}(t)/|x_\mu-t|^{n+\beta})\,dt\leqq C[\text{diam }(K_\mu)]^\alpha$ by noting that $\Delta^{k+1}(t)\leqq|x_\mu-t|$ and that diam $(S_\mu)\leqq C$ diam K_μ. Also, since $|x|\geqq|x_\mu|/16$, one can see that

$$\int_{K_\mu}\frac{dx}{|x|^{n+\alpha}}\int_{S'_\mu}\frac{\Delta^{k+1}(t)}{|x_\mu-t|^{n+\beta}}\,dt$$

is majorized by a constant times $[\text{diam }K_\mu]^{n+\alpha}\cdot|x_\mu|^{-(n+\alpha)}$. Hence by the Lemma 5″ the contribution of (11) on the S'_μ is finite and we only need to consider the contribution over the t with $|x|/16\leqq|t|\leqq 16|x|$ outside the S'_μ. For such t, $|t-x_\mu|\geqq\frac{1}{2}|t-x|$ since $x\in K_\mu\subset S_\mu$. Let ψ_Q be the characteristic function of Q and let $\lambda_x(t)$ be equal to zero if $t\in S'_\mu$ and $x\in K_\mu$, and let $\lambda_x(t)$ be one otherwise. Then the part of (11) that remains to be considered is a constant multiple of

$$\int_Q\frac{dx}{|x|^{n+\alpha}}\left\{\int_{|x|/16\leqq|t|\leqq 16|x|}\frac{\Delta^{k+1}(t)\lambda_x(t)}{|t-x|^{n+\beta}}\,dt\right\}$$
(12)
$$\leqq C\int_{E_n}\frac{\Delta^{k+1}(t)}{|t|^{n+\alpha}}\,dt\int_{|t|/16\leqq|x|\leqq 16|t|}\frac{\psi_Q(x)\lambda_x(t)}{|t-x|^{n+\beta}}\,dx.$$

Now the inner integral is taken over the exterior of Q to the sphere S'_μ which contains x in K_μ, i.e., it is

$$\leqq C\int_{2|x-t|\geqq\Delta(t)}\frac{dx}{|t-x|^{n+\beta}}\leqq C\Delta^{-\beta}(t).$$

Combining this with the above we have that (12) is dominated by

$$C\int_{|t|\leqq\infty}\frac{\Delta^{k+1-\beta}(t)}{|t|^{n+\beta}}\,dt = C\int\frac{\Delta^\alpha(t)}{|t|^{n+\alpha}}\,dt.$$

This is finite by Lemma 5.

It now remains to show that (9) is finite. To do this recall that h satisfies the condition Λ^2_α uniformly in the set P, i.e., we can write $f(x_0+t)=P_{x_0}(t)+\rho_{x_0}(t)$ where $P_{x_0}(t)$ is a polynomial of degree $\leqq k$ and $\int_{|t|\leqq\rho}|\rho_{x_0}(t)|^2\,dt=O(\rho^{2\alpha+n})$, as $\rho\to 0$, uniformly for x_0 in P. We can apply Lemma 6 to h to see that for x_0 the origin we have $h(x)=P(x)+\rho(x)$ as in (7) and if we write $x_\mu+t=x$ we have $h(x)=\sum_{|j|=0}^k(h_j(x_\mu)t^j/j!)+\rho_{x_\mu}(x-x_\mu)$. For x in P $(\partial/\partial x)^jP(x)=h_j(x)$ and hence we have $h_j(x_\mu)-(\partial/\partial x)^j P(x_\mu)=O(|x_\mu|^{\alpha-|j|})$. Also we have that $P(x)=\sum_{|j|=0}^{k}(\partial/\partial x)^j \times P(x_\mu)(x-x_\mu)^j/j!$. Combining all these we can write

$$\rho(x)-\rho(x_\mu) = h(x)-h(x_\mu)-\{P(x)-P(x_\mu)\}$$
$$= \sum_{|j|=1}^{k}\frac{h_j(x_\mu)}{j!}t^j+\rho_{x_\mu}(x-x_\mu)-\sum_{|j|=1}^{k}\frac{p_j(x_\mu)(x-x_\mu)^j}{j!}.$$

Hence

$$|\rho(x)-\rho(x_\mu)| \leqq \rho_{x_\mu}(x-x_\mu)+O\left(\sum_{|j|=1}^{k}|x_\mu|^{\alpha-|j|}|x-x_\mu|^{|j|}\right).$$

Using this to estimate (9) we see that with the definition of h in Λ_α^2

$$\sum_\mu \frac{[\rho(x)-\rho(x_\mu)]^2}{|x|^{2\alpha+n}}dx \leqq C\sum_\mu \frac{[\operatorname{diam}(K_\mu)]^{n+2\alpha}}{|x_\mu|^{n+2\alpha}}+C\sum_\mu \frac{[\operatorname{diam}(K_\mu)]^{n+2}}{|x_\mu|^{n+2}}.$$

The last inequalities are demonstrated by recalling the inequalities involving x and x_μ where x is in K_μ. (9) is finite by Lemma 5″.

As a final remark we point out that the condition N_α^p easily implies the condition Λ_α^p which is enough to complete the proof of Theorem 2 and Theorem 2′.

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